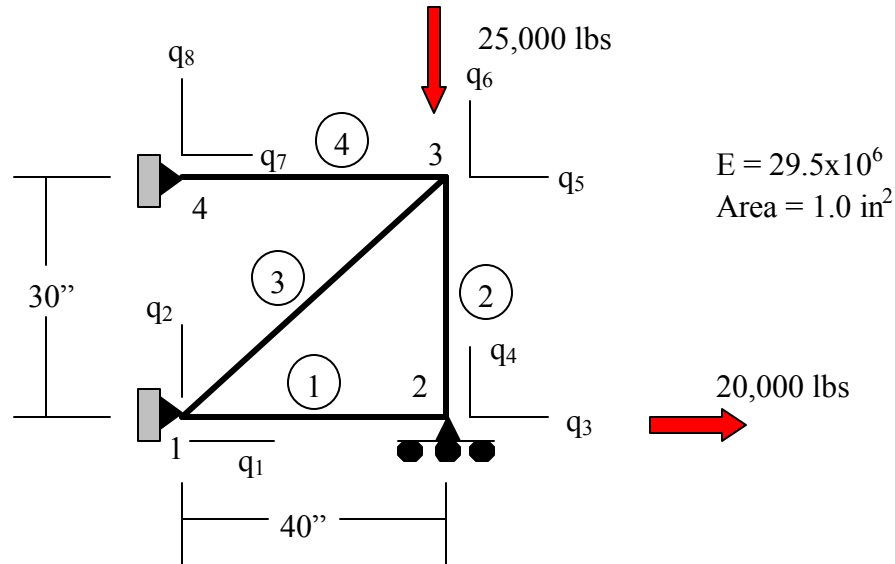# More Finite Element Truss Problem

7.0   Truss Example

In the last lecture we looked at using a finite element technique to computer the stresses in a truss. We also developed the global stiffness matrix. We will use the methods develop there to solve a problem. Consider

7.1   Computing Nodal Displacements

There are 4 nodes and 4 elements making up the truss. We are going to do a two dimensional analysis so each node is constrained to move in only the X or Y direction. We call these directions of motion degrees of freedom or dof for short. There are 4 nodes and 8 degrees of freedom (two degrees of freedom for each node). We can number the degrees of freedom with the formulas:

Vertical degree of freedom $dof = 2*node$ (7.1)
Horizontal degree of freedom $dof = 2*node-1$ (7.2)

where *node* is the node number.

We can locate each node by its coordinates. The table below shows the coordinates of the nodes in the problem we are solving. We can use these coordinates to determine the lengths and angles of the elements.

Node	X	Y
1	0	0
2	40	0
3	40	30
4	0	30

Each element can be described as extending from one node to another. This also can be defined in a table.

Element	From Node	To Node
1	1	2
2	3	2
3	1	3
4	4	3

From these two tables we can derive the lengths of each element and the cosine and sine of their orientation. This is shown in the table below.

Element	Length	Cosine	Sine
1	40	1	0
2	30	0	-1
3	50	0.8	0.6
4	40	1	0

In the last lecture we developed the stiffness matrix for an element.

$$k = \frac{AE}{L} \begin{bmatrix} c^2 & cs & -c^2 & -cs \\ cs & s^2 & -cs & -s^2 \\ -c^2 & -cs & c^2 & cs \\ -cs & -s^2 & cs & s^2 \end{bmatrix} \tag{6.21}$$

This stiffness matrix is for an element. The element attaches to two nodes and each of these nodes has two degrees of freedom. The rows and columns of the stiffness matrix correlate to those degrees of freedom.

Using the equation shown I (6.21) we can construct that stiffness matrix for element 1 defined in the table above. The stiffness matrix is:

Global dof

$$k_1 = \frac{29.5x10^6}{40} \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ -1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \tag{7.3}$$

Element 2

$$
k_2 = \frac{29.5x10^6}{30}
\begin{matrix}
5 & 6 & 3 & 4 & \\
\begin{bmatrix}
0 & 0 & 0 & 0 \\
0 & 1 & 0 & -1 \\
0 & 0 & 0 & 0 \\
0 & -1 & 0 & 1
\end{bmatrix}
\begin{matrix}
5 \\ 6 \\ 3 \\ 4
\end{matrix}
\end{matrix}
\tag{7.4}
$$

Element 3

$$
k_3 = \frac{29.5x10^6}{50}
\begin{matrix}
1 & 2 & 5 & 6 & \\
\begin{bmatrix}
.64 & .48 & -.64 & -.48 \\
.48 & .36 & -.48 & -.36 \\
.64 & .48 & .64 & .48 \\
-.48 & -.36 & .48 & .36
\end{bmatrix}
\begin{matrix}
1 \\ 2 \\ 5 \\ 6
\end{matrix}
\end{matrix}
\tag{7.5}
$$

Element 4

$$
k_4 = \frac{29.5x10^6}{40}
\begin{matrix}
7 & 8 & 5 & 6 & \\
\begin{bmatrix}
1 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{matrix}
7 \\ 8 \\ 5 \\ 6
\end{matrix}
\end{matrix}
\tag{7.6}
$$

The next step is to add the stiffness matrices for the elements to create a matrix for the entire structure. We can facilitate this by creating a common factor for Young's modulus and the length of the elements.

For element 1, we divide the outside by 15 and multiply each element of the matrix by 15. Multiplying and dividing by the same number is the same as multiplying and dividing by 1.

$$
k_1 = \frac{29.5x10^6}{600}
\begin{matrix}
1 & 2 & 3 & 4 & \\
\begin{bmatrix}
15 & 0 & -15 & 0 \\
0 & 0 & 0 & 0 \\
-15 & 0 & 15 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}
\begin{matrix}
1 \\ 2 \\ 3 \\ 4
\end{matrix}
\end{matrix}
\tag{7.7}
$$

We multiply and divide element 2 by 20.

$$k_2 = \frac{29.5x10^6}{600} \begin{array}{cccc} 5 & 6 & 3 & 4 \\ \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 20 & 0 & -20 \\ 0 & 0 & 0 & 0 \\ 0 & -20 & 0 & 20 \end{bmatrix} & \begin{array}{c} 5 \\ 6 \\ 3 \\ 4 \end{array} \end{array} \tag{7.8}$$

Multiply and divide element 3 by 12.

$$k_3 = \frac{29.5x10^6}{600} \begin{array}{cccc} 1 & 2 & 5 & 6 \\ \begin{bmatrix} 7.68 & 5.76 & -7.68 & -5.76 \\ 5.76 & 4.32 & -5.76 & -4.32 \\ 7.68 & 5.76 & 7.68 & 5.76 \\ -5.76 & -4.32 & 5.76 & 4.32 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 5 \\ 6 \end{array} \end{array} \tag{7.9}$$

We do the same for element 4 by multiplying and dividing it by 15.

$$k_4 = \frac{29.5x10^6}{600} \begin{array}{cccc} 7 & 8 & 5 & 6 \\ \begin{bmatrix} 15 & 0 & -15 & 0 \\ 0 & 0 & 0 & 0 \\ -15 & 0 & 15 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 7 \\ 8 \\ 5 \\ 6 \end{array} \end{array} \tag{7.10}$$

The coefficient for each stiffness matrix is the same so we can easily add the matrices. We add the degree of freedom for each element stiffness matrix into the same degree of freedom in the structural matrix. The resulting structural stiffness matrix is shown below.

$$K = \frac{29.5 \times 10^6}{600} \begin{array}{cccccccc} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15 & 0 \\ -5.76 & 4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} & \begin{array}{c} 1 \\ 2 \\ 3 \\ 4 \\ 5 \\ 6 \\ 7 \\ 8 \end{array} \end{array} \tag{7.11}$$

Remembering our basic equation

$$KQ = F \tag{7.12}$$

where K is the structural stiffness matrix, Q is the displacement of each node, and F is the external force matrix. This results in

$$\frac{29.5\times10^6}{600}\begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ -15.0 & 0 & 15.0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 20.0 & 0 & -20.0 & 0 & 0 \\ -7.68 & -5.76 & 0 & 0 & 22.68 & 5.76 & -15 & 0 \\ -5.76 & 4.32 & 0 & -20.0 & 5.76 & 24.32 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix}\begin{bmatrix} q_1 \\ q_2 \\ q_3 \\ q_4 \\ q_5 \\ q_6 \\ q_7 \\ q_8 \end{bmatrix}=\begin{Bmatrix} 0 \\ 0 \\ 20{,}000 \\ 0 \\ 0 \\ -25{,}000 \\ 0 \\ 0 \end{Bmatrix} \tag{7.13}$$

We have boundary conditions at the fixed supports. Our assumption is that these joints will not move in the constrained direction. We remove these from our matrix. The constrained displacements are dof 1, 2, 4, 7, and 8. The resulting matrix is:

$$\frac{29.5\times10^6}{600}\begin{bmatrix} 15 & 0 & 0 \\ 0 & 22.68 & 5.76 \\ 0 & 5.76 & 24.32 \end{bmatrix}\begin{Bmatrix} q_3 \\ q_5 \\ q_6 \end{Bmatrix}=\begin{Bmatrix} 20{,}000 \\ 0 \\ -25{,}000 \end{Bmatrix} \tag{7.14}$$

We can use Gaussian elimination or any number of other solution techniques to solve the system of equations shown above. Doing so yields

$$\begin{Bmatrix} q_3 \\ q_5 \\ q_6 \end{Bmatrix}=\begin{Bmatrix} 27.12\times10^{-3} \\ 5.65\times10^{-3} \\ -22.25\times10^{-3} \end{Bmatrix} \quad \text{inches} \tag{7.15}$$

7.2   Computing Stresses

From a previous lecture we know that

$$\boldsymbol{s} = \frac{E}{L}\{-c \quad -s \quad c \quad s\}\overline{q} \tag{6.28}$$

We use this equation to compute the stress in each element.

$$s_1 = \frac{29.5\times10^6}{40}\{-1 \quad 0 \quad 1 \quad 0\}\begin{Bmatrix} 0 \\ 0 \\ 27.12\times10^{-3} \\ 0 \end{Bmatrix}\begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} \overset{\text{dof}}{\diagup} \tag{7.16}$$

or

$$s_1 = 20{,}000\, psi \tag{7.17}$$

$$s_2 = \frac{29.5\times10^6}{30}\{0 \quad 1 \quad 0 \quad -1\}\begin{Bmatrix} 5.65\times10^{-3} \\ -22.25\times10^{-3} \\ -27.12\times10^{-1} \\ 0 \end{Bmatrix}\begin{matrix} 5 \\ 6 \\ 3 \\ 4 \end{matrix} \tag{7.18}$$

$$s_2 = 21{,}880.0\, psi \tag{7.19}$$

Using a similar technique we get

$$s_3 = 5{,}208\, psi \tag{7.20}$$

and

$$s_4 = 4{,}167\, psi \tag{7.21}$$

7.3   Computing the Reactions

The last step is to compute the support reactions. We need to determine the reaction forces along dof 1, 2, 3, 7, and 8 which correspond to the fixed supports. These are obtained by substituting Q into the original finite element equation.

$$R = KQ - F \tag{7.22}$$

We only need to use those rows of the structural stiffness matrix that correspond to the fixed supports. At these supports, we are not supplying an external force so F=0. Our equation becomes

$$R = KQ \tag{7.23}$$

or

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \frac{29.5 \times 10^6}{600} \begin{bmatrix} 22.68 & 5.76 & -15.0 & 0 & -7.68 & -5.76 & 0 & 0 \\ 5.76 & 4.32 & 0 & 0 & -5.76 & -4.32 & 0 & 0 \\ 0 & 0 & 0 & 20 & 0 & -20 & 0 & 0 \\ 0 & 0 & 0 & 0 & -15.0 & 0 & 15.0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \begin{Bmatrix} 0 \\ 0 \\ 27.12 \times 10^{-3} \\ 0 \\ 5.65 \times 10^{-3} \\ -22.25 \times 10^{-3} \\ 0 \\ 0 \end{Bmatrix} \quad (7.23)$$

We multiply the stiffness matrix K and the deformation vector Q to get the reactions. They are shown in the following equation.

$$\begin{Bmatrix} R_1 \\ R_2 \\ R_4 \\ R_7 \\ R_8 \end{Bmatrix} = \begin{Bmatrix} -15,8333 \\ 3,126 \\ 21,879 \\ -4,167 \\ 0 \end{Bmatrix} \quad (7.24)$$